Journal of Air Transport Management 17 (2011) 142–146

Contents lists available at ScienceDirect

Journal of Air Transport Management

journal homepage: www.elsevier.com/locate/jairtraman

Note

Airline overbooking models with misspecification

Kannapha Amaruchkul*, Patipan Sae-Lim

School of Applied Statistics, National Institute of Development Administration (NIDA), 118 Serithai Road, Bangkapi, Bangkok 10240, Thailand

Keywords:
Revenue management
Overbooking
Stochastic model applications

A B S T R A C T

This paper looks at static overbooking models. In theory, a random show demand follows a binomial distribution with each reservation showing up independently and with the same probability. However, in practice, some overbooking models assume that the show demand is the product of the overbooking level and the random show-up rate. The decision model embedded in a commercial revenue management system is misspecified. We explore the consequences of the modeling error and find that the performance of the model with misspecification decreases as show-up probability decreases. Among our three choices of show-up rate distributions, normal, beta and deterministic, the beta model performs best. We also identify situations in which an airline might prefer the deterministic model to the normal model.

1. Introduction

Overbooking is practiced by nearly all passenger airlines. They accept more reservations than their fixed capacity to compensate for cancellations and no-shows, which can be as high as 50% (Smith et al., 1992). The financial gain from the overbooking practice has been estimated to be at least $1 billion (Bailey, 2007). The main objective of the approach is to find an overbooking level/limit – the maximum number of reservations to hold at any time–that minimizes an expected cost. This cost is calculated with respect to the probability distribution of the show demand (show-ups), the number of reservations that survive to the time of services. The cost is composed of an oversale cost, which occurs if the realized show demand exceeds the capacity, and a spoilage cost, which occurs if the realized show demand is less than the capacity.

The functional form of the show-ups can affect the overbooking level recommended implied; models with different specifications of the show demand can lead to different expected costs, and may yield different overbooking levels. Models commonly assume that the show demand is linear in the overbooking level; i.e., given the overbooking level x, the show demand is xR, where the random variable R is the show-up rate. In some revenue management (RM) systems, this rate is modeled using a normal/Gaussian distribution with the mean and variance are periodically estimated from historical data.

The specification with the show demand equaling the product of the overbooking level and the show-up rate is simple and widely used in practice. It is, however, theoretically deficient under certain conditions. Suppose that, each reservation shows up independently, and that the probability of showing up is identical among all reservations. In this case, the show demand given the overbooking level x follows a binomial distribution with parameters x and θ, where θ is the show-up probability. Under these conditions the linear assumption in the airline's decision model is incorrect – there is a model misspecification occurs.

In the RM, there is an iterative process in which the control (e.g., the overbooking level) from the optimization model is enacted, the data (e.g., the realized show-ups) are collected over several flights, the parameters (e.g., the mean and variance of the show-up rate distribution) are forecasted and, finally, the new control is determined from the optimization model, given the updated parameters. In this article, we want to explore the consequences of the modeling error from which the optimization model is misspecified.

Using the optimization model, we consider three show-up rate distributions, namely normal, beta and deterministic. For each of the three misspecified models, we provide a closed-form expression for the overbooking level. To benchmark and evaluate these models, we construct a model in which the show demand theoretically follows a binomial distribution. We also obtain an optimal overbooking level with respect to the benchmark (binomial) model. To study the behavior of the iterative process with the misspecified optimization model, we perform a series of numerical experiments. We find that as the iterative process goes on for a long time, the sequence of the average costs with the given misspecified model converges almost everywhere. The long-run average cost is greater than the optimal expected cost with the binomial model. In all tested problem instances, the beta model performs best; the

* Corresponding author.
E-mail address: kamaruchkul@gmail.com (K. Amaruchkul).

0969-6997/$ – see front matter
doi:10.1016/j.jairtraman.2010.06.003

percent difference between the optimal cost and the estimated cost using the beta model is at most 10%. This suggests that the beta model is quite robust to misspecification. Moreover, we identify situations in which the performance of the deterministic model and that of the normal model are approximately the same.

Overbooking models are of two broad types: dynamic and static. Regarding the former, the dynamics of reservation requests and customer cancellations over time are ignored, whereas in dynamic-models, such inter-temporal effects are explicitly accounted for. We consider only the static overbooking model because it is similar to the model used in most commercial RM systems. The classic static overbooking model assumes that the show demand follows a binomial distribution (Talluri and van Ryzin, 2004), e.g. Thompson (1961) finds that this adequately fits the data collected from Tasman Empire Airways. Other static models assume that the show demand is the product of the overbooking level and the show-up rate. The random show-up rate can be modeled using a parametric distribution, such as uniform (Kasilingam, 1997), beta (Luo et al., 2009) or normal. Popescu et al. (2006) argue that modeling the show-up rate as a normal random variable, as is quite common in practice, is not appropriate and they use a nonparametric method to obtain a histogram, in which the number and size of bins are constructed based on a wavelet method. While these studies look at static overbooking problems alone, without any iterative process, the parameters of the show-up rate distribution developed here are iteratively updated.

2. Overbooking

Because of its simplicity, the static overbooking problem is the basis of the most widely used methodology for making overbooking decisions. An overbooking problem is defined as determining an overbooking level such that expected costs is minimized. Since an airline operates many repeat flights, we can assume that decision maker is risk neutral, and the objective of minimizing the expected value is appropriate.

Let $\mathbb{N}$ be the set of natural numbers and Z the set of integers. For a real number y, $[y]^+ = \max(y, 0)$ the positive part of y, $\lfloor y \rfloor = \max\{n \in Z : n \le y\}$ the floor of y and $\lceil y \rceil = \min\{n \in Z : n \ge y\}$ the ceiling of y. Assume that the capacity is a known constant c. If an overbooking level is set to x, denote the random show demand as $S(x)$. Let $a_o \ge 0$ be the per-unit oversale cost, and $a_s \ge 0$ the per-unit spoilage cost. The expected cost is thus:

$$\tilde{f}(x) = E\Big[a_o[S(x) - c]^+ + a_s[c - S(x)]^+\Big] \quad (1)$$

The first and second terms are the oversale and spoilage costs, respectively. The over salecost is computed as the per-unit oversale cost a_o times the number of show-ups that are denied boarding $[S(x) - c]^+$. The spoilage cost is found similarly. Consider the following problem:

$$\min\Big\{f(x) = (a_o + a_s)E\Big[(S(x) - c)^+\Big] - a_s E[S(x)]\Big\} \quad (2)$$

Since $\tilde{f}(x) = f(x) + a_s c$, an optimal overbooking level that minimizes $\tilde{f}(x)$ is identical to the one that minimizes $f(x)$. Eq. (2) is the basis of the study.

The airline chooses an overbooking level that minimizes the expected cost, which is calculated with respect to the distribution of the show demand $S(x)$. In practice, it is usual that the airline does not know the distribution of the show demand, but it makes overbooking decisions based on perceived models. To evaluate and benchmark the perceived models, we develop the actual model, in which the distribution of the show demand is known. We adopt terminology similar to that used in Evans and Honkapohja (2001).

2.1. Actual model

Suppose that $x \in N$ reservations have been made in advance and that each requires a single seat (i.e. there are no group bookings). It is assumed that reservations show up independently and that the probability of showing up is identical for all reservations. With the actual model, the show demand $S_0(x)$ follows a binomial distribution with parameters x and θ. Let $f_0(x)$ be the objective function in Eq. (2), where we replace $S(x) = S_o(x)$.

Proposition 1. *With the actual model, the objective function $f_0(x)$ is convex on $\mathbb{N}$. An optimal overbooking level is given as*

$$x_0^* = \operatorname{argmax}\{x \in \mathbb{N} : (a_s + a_o)P(S_0(x-1) \ge c) \le a_s\} \quad (3)$$

The optimality condition can be explained intuitively as follows. Given that the current overbooking level is $x - 1$, we want to know whether to overbook one or more seat. Oversale cost is incurred if the show demand from the current reservations [of $(x - 1)$ seats] is at least the capacity. Hence, the expected marginal oversale cost is $a_o P(S_0(x-1) \ge c)$. We would incur a spoilage cost, if the show demand from the current reservations is less than the capacity. Thus, the expected marginal spoilage cost is $a_s P(S_0(x-1) < c)$. If the expected marginal spoilage cost is at least the expected marginal oversale cost [i.e., $a_o P(S_0(x-1) \ge c) \le a_s P(S_0(x-1) < c)$], then we would overbook one more seat.

2.2. Perceived models

Popescu et al. (2006), Kasilingam (1996) and others suggest that airlines typically do not use a sophisticated approach to predict the show demand. It is assumed that the show demand is linear in the overbooking level. Specifically, if the overbooking level is equal to x, then the show demand is xR, where R is the show-up rate. The distribution of the show-up rate is constructed from historical data.

We look at parametric methods and consider three distributions that an airline might use to model the show-up rate. Let R_i be the random show-up rate in the perceived model $i \in \{1, 2, 3\}$ where R_1 has a degenerate distribution; i.e., $P(R_1 = \rho) = 1$, with $\rho \in (0, 1)$ representing a deterministic show-up rate. R_2 has a normal distribution with mean μ and variance σ^2, and R_3 follows a standard beta distribution with shape parameters a and b. Suppose that the airline uses perceived model i. Let y_i denote the parameter of the show-up distribution R_i; i.e., $y_1 = \rho, y_2 = (\mu, \sigma^2), y_3 = (a, b)$. Let $f_i(\cdot|y_i)$ be the objective function in (2), where $S(x) = xR_i$. Denote $\tilde{x}_i(y_i) = \operatorname{argmin} f_i(x|y_i)$. Let $g_i(\cdot|y_i)$ be the probability density function of R_i for each $i = \{2, 3\}$.

Proposition 2. *For each i, the objective function $f_i(x|y_i)$ is convex in x. With perceived model 1, $\tilde{x}_1(y_1) = c/y_1$. With perceived model $i \in \{2, 3\}$, $\tilde{x}_i(y_i)$ solves*

$$\int_{\alpha_i}^{c/\tilde{x}_i(y_i)} t g_i(t|y_i) \mathrm{d}t = \frac{a_o}{(a_s + a_o)} E[R_i] \quad (4)$$

where $\alpha_2 = -\infty$, and $\alpha_3 = 0$.

The left-hand side in Eq. (4) is decreasing in $\tilde{x}_i(y_i)$. The overbooking level can be found via a classical search procedure. The solution $\tilde{x}_i(y_i)$ found in the proposition may not be an integer, but if an integer overbooking level is desired, one could set it to $\tilde{x}_i^*(y_i) = \operatorname{argmin}\{f_i(\lfloor \tilde{x}_i(y_i) \rfloor \, |y_i), f_i(\lceil \tilde{x}_i(y_i) \rceil \, | \, y_i)\}$.

3. Iterative process

Suppose that the airline operates many repeat flights. With each perceived model, an optimal overbooking level depends on the show-up rate distribution, whose parameters are periodically forecasted from the historical data. As new data become available, the airline updates the parameters of the show-up rate distribution, the overbooking level is chosen with respect to the updated distribution, and the process continues. These are sometimes referred to as the iterative data collection-forecasting-optimization process.

Suppose that the airline updates information every m flights. Define the t-th decision period to be a time in which the t-th forecast becomes available. Each period of the process consists of three steps: optimization, data collection, and forecasting, respectively. At the beginning of the t-th period, the forecasts for three perceived models are $y_{1t} = \rho_t,\ y_{2t} = (\mu_t, \sigma_t^2),\ y_{3t} = (a_t, b_t)$. In the optimization step, the overbooking level with model i is $x_i^*(y_{it})$. In the data-collection step, the airline with perceived model i, realizes m show demands $\{s_{ijt} : j = 1, \ldots, m\}$, the random sample from the actual distribution [the binomial distribution with parameters $x_{it}^*(y_{it})$ and θ]. The realized show-up rates are $\{r_{ijt} : j = 1, \ldots, m\}$ where $r_{ijt} = s_{ijt}/x_{it}^*(y_{it})$. In the forecasting step, the show-up rate for the next decision period is forecasted based on the simple exponential smoothing technique: $y_{i,t+1} = \gamma_i \hat{r}_{it} + (1-\gamma_i) y_{it}$ where $\gamma_i \in (0,1)$ is the smoothing parameter, and $\hat{r}_{it}$ is the maximum likelihood estimator (MLE) of the parameters of the show-up rate distribution, determined from the realized show-up rate $\{r_{ijt} : j = 1, \ldots, m\}$. For instance, with perceived model 1, $\hat{r}_{it} = \bar{r}_{1t}$ and perceived model 2, $\hat{r}_{2t} = (\bar{r}_{2t}, (\sum_{j=1}^m (r_{2jt} - \bar{r}_{2t})^2)/m)$, where $\bar{r}_{it} = (\sum_{j=1}^m r_{ijt})/m$. We restrict ourselves to the exponential smoothing method, because of its simplicity and popularity (Makridakis et al., 1998), and to the MLE. After obtaining the new forecast $y_{i,t+1}$, the process continues with the optimization step in period $(t+1)$ to determine $x_{i,t+1}^*(y_{i,t+1})$ etc.

4. Numerical experiments

Two sets of numerical experiments are conducted. In the first, we study the asymptotic behavior of the perceived model as the number of decision periods in the iterative process becomes very large. In the second, we compare the per-flight costs if the airline implements the overbooking level from each of the three perceived models. To estimate the expected costs, we perform a Monte Carlo simulation. With the perceived models, the parameters of the show-up rate distribution are updated every $m = 30$ flights, and the smoothing parameters are $\gamma_1 = \gamma_2 = \gamma_3 = 0.5$.

4.1. Asymptotic behavior investigation

Data for the experiments are obtained from one of the leading passenger airlines in Thailand. We consider a single-leg weekly flight with capacity $c = 338$ seats. The airline sets the per-unit oversale and spoilage costs to $a_s = a_o = 4800$, which is the reference fare of the flight. With the actual model, given an overbooking level x, the show demand follows the binomial distribution with parameters x and $\theta = 0.945$.

We report only on the deterministic model, perceived model 1, because the asymptotic behaviors of the other models are similar. Fig. 1 shows four sample paths, when the initial show-up rates are 0.945, 0.945, 0.875 and 0.875. From Fig. 1a, the sequence of the overbooking levels does not converge. For instance, when the initial show-up rate is $\rho_1 = 0$ .945, the last five overbooking levels of the first sample path (the solid line) are 357, 357, 357, 357, 358, whereas those of the second sample path (the dotted line) are 357, 358, 358, 357, 358.

Fig. 1b suggests that two sample paths of the average costs corresponding to $\rho_1 = 0.945$ converge, and so do the other two corresponding to $\rho_1 = 0.875$. Moreover, all four sample paths of the average costs converge to a single number; approximately

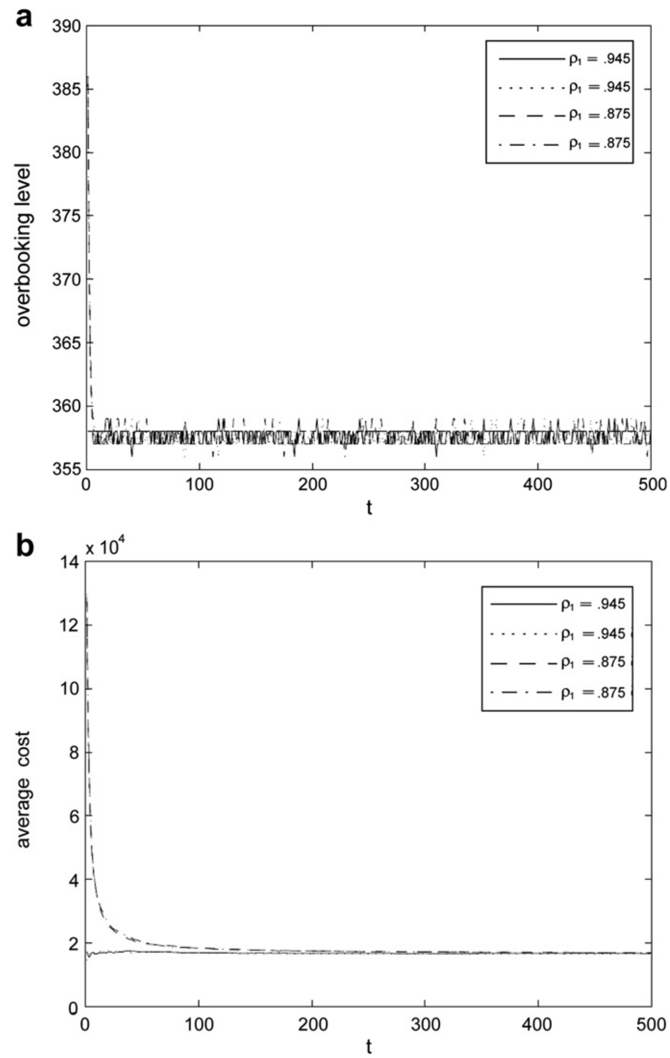

Fig. 1. Some sample paths with perceived model 1. (a) Overbooking levels. (b) Average costs.

16,600. With different show-up rates, the sequences of the average costs do converge to the same point. When many replications are carried out, the results suggest that almost all sample paths converge to a single point. We conclude that the long-run average cost converges to a constant with probability one. Although it does not converge to the optimal cost based on the actual model (which is 16522.26), the difference between the two is only 0.47%.

4.2. Performance evaluation

In the second set of experiments, let $m = 30$, and $c = 338$ (as in the first set). Let the initial forecasts for the three perceived models be $\rho_1 = 0.945$, $(\mu_1, \sigma_1^2) = (0.945, 0.026^2)$ and $(a_1, b_1) = (62.327, 3.855)$. The rest of the problem parameters – the show-up probability θ and the pair of oversale and spoilage costs (a_s, a_o) – are varied systematically according to the 3×4 factorial experiment. The first factor, the show-up probability, has three levels $\theta\, \varepsilon$ {0.8, 0.5, 0.3}, and the second factor, the cost parameter, has four levels $(a_s, a_o)\, \varepsilon$ {(4800, 4800), (4800, 9600), (9600, 4800), (9600, 9600)}. The problem parameters for the experiments are shown in Table 1a.

Table 1
Performance of perceived models.

(a) Problem parameters for all twelve experiments

$\theta \backslash (a_s, a_o)$	(4800, 4800)	(4800, 9600)	(9600, 4800)	(9600, 9600)
0.8	Ex.1	Ex.2	Ex.3	Ex.4
0.5	Ex.5	Ex.6	Ex.7	Ex.8
0.3	Ex.9	Ex.10	Ex.11	Ex.12

(b) Cost associated with overbooking levels from perceived models

Experiment	Actual model optimal cost	Linear models								
		Perceived model 1			Perceived model 2			Perceived model 3		
		Mean	% Diff.	SE	Mean	% Diff.	SE	Mean	% Diff.	SE
1	31,456.0	32,577.5	3.6	35.0	32,598.6	3.6	34.9	32,252.0	2.5	31.8
2	42,542.3	47,877.6	12.5	32.4	43,879.9	3.1	36.5	43,502.3	2.3	32.5
3	43,458.6	49,049.9	12.9	73.2	45,449.5	4.6	65.6	44,810.5	3.1	60.7
4	62,912.1	65,154.3	3.6	69.9	65,202.2	3.6	69.8	64,508.5	2.5	63.7
5	49,769.6	53,986.8	8.5	127.4	53,989.9	8.5	127.5	52,264.6	5.0	105.5
6	67,473.2	78,414.3	16.2	121.6	71,840.8	6.5	127.1	70,063.2	3.8	103.8
7	68,595.1	82,946.1	20.9	261.1	76,698.9	11.8	253.9	73,493.6	7.1	212.3
8	99,539.2	107,988.8	8.5	255.1	108,001.2	8.5	255.3	104,498.8	5.0	210.7
9	58,884.4	66,244.9	12.5	206.1	66,270.6	12.5	206.5	62,749.0	6.6	158.8
10	79,863.3	95,203.0	19.2	198.8	87,461.9	9.5	205.2	84,047.1	5.2	158.2
11	81,140.6	103342.7	27.4	421.4	95,679.5	17.9	415.0	88,700.2	9.3	319.1
12	117,768.9	132,546.4	12.5	412.9	132,619.4	12.6	413.7	125,721.7	6.8	319.1

In each simulation, we fix the number of decision periods to be 400. (From Fig. 1b, the average cost for a given initial show-up rate has already settled down since the 400-th decision period.) Table 1b shows the optimal expected cost based on the actual model, the estimated cost when the airline uses the overbooking level from the perceived model, the percent difference between the two costs, and the standard error of the estimated cost. In each experiment, the number of simulation replications is 300. (In all experiments in Table 1b, the standard error is within 0.5% of the estimated cost.)

In each of the experiments, the beta model, perceived model 3, outperforms the deterministic and normal models. The percent difference between the estimated cost with the beta model and the optimal expected cost ranges from two to nine. The beta show-up rate is quite robust to modeling error. Moreover, in most cases the estimated variance of the cost using the beta model is the smallest. The beta model is "better" than the other two linear models, regardless of whether the objective is to minimize the expected cost or to minimize the variance of the cost.

In each of the six experiments in which $a_s \neq a_o$, the estimated cost with the deterministic model (perceived model 1) is highest among the three perceived models. Recall that the deterministic model takes into account only the forecasted show-up rate, whereas the other perceived models take into account not only the forecast but also the cost parameters. Thus showing that the indicating the deterministic model performs the worst is not surprising.

In the six experiments, the normal model is outperformed by the beta model. This may result from the unbounded tail of the normal distribution. Suppose that the overbooking level is $x \in \mathbb{N}$. The number of show demand in the benchmark model $S_0(x)$ is binomially distributed with parameters x and θ, whereas that in perceived model $i \in \{2, 3\}$ is $S_i(x) = xR_i$ where random variables R_2 and R_3 are normal and beta, respectively. In the benchmark model, $S_0(x)$ has a support on $\{0, 1, 2, \ldots, x\}$; the actual show demand $S_0(x)$ never exceeds the overbooking level x. Recall that the support of the beta distribution is $(0, 1)$, whereas that of the normal distribution is $(-\infty, \infty)$. The show demand in the normal model can exceed the overbooking level. There is a strictly positive probability that the show demand in normal model is greater than the overbooking level; i.e., $P(S_2(x) > x) = P(R_2 > 1) = 1 - \Phi((1-\mu)/\sigma) > 0$ where μ and σ^2 are mean and variance of R_2, respectively, and Φ is the distribution function of the standard normal random variable. In the beta model, such probability is zero; i.e., $P(S_3(x) > x) = P(R_3 > 1) = 0$. In each of the benchmark and beta models, the show demand cannot exceed the overbooking level, but in the normal model it can. The beta model is more theoretically sound than the normal model.

In each of the other experiments in which $a_s = a_o$, the deterministic and normal models perform about the same. Recall that in the deterministic model the overbooking level is in a simple closed-form, whereas in the normal model it is found numerically (Proposition 2).[1]

From Table 1b, the percent difference increases as the show-up probability decreases. For instance, with $(a_s, a_o) = (4800, 4800)$ and with perceived model 3, the percent difference increases from 2 to 5 to 7, when the show-up probability decreases from 0.8 to 0.5 to 0.3. If the airline anticipates a high show-up probability, then making an overbooking decision with the perceived model might be acceptable. However, if it anticipates a low show-up probability, the airline needs to be very cautious using the overbooking level recommended by the perceived model, since the performance of the perceived model gets worse when the show-up probability decreases.

Acknowledgments

The authors would like to thank referees of the journal for their helpful comments and suggestions. Patipan is a master student supported by a graduate scholarship from NIDA, Thailand.

References

Bailey, J., May 30, 2007. Bumped fliers and no plan B. The New York Times.

Evans, G.W., Honkapohja, S., 2001. Learning and Expectations in Macroeconomics. Princeton University Press, Princeton.

[1] The computational time of the deterministic model is much less than that of the normal model. Hence, the deterministic model might be preferred to the normal model for some airlines with $a_s = a_o$.

Kasilingam, R.G., 1996. Air cargo revenue management: characteristics and complexities. European Journal of Operational Research 96, 36–44.

Kasilingam, R.G., 1997. An economic model for air cargo overbooking under stochastic capacity. Computers and Industrial Engineering 32, 221–226.

Luo, S., Akanyildrim, M.C., Kasilingam, R.G., 2009. Two-dimensional cargo overbooking models. European Journal of Operational Research 197, 862–883.

Makridakis, S., Wheelwright, S.C., Hyndman, R.J., 1998. Forecasting: Methods and Applications. John Wiley & Sons, New York.

Popescu, A., Keskinocak, P., Johnson, E., LaDue, M., Kasilingam, R., 2006. Estimating air-cargo overbooking based on a discrete show-up-rate distribution. Interfaces 36, 248–258.

Smith, B.C., Leimkuhler, J.F., Darrow, R.M., 1992. Yield management at American airlines. Interfaces 22, 8–31.

Talluri, K., van Ryzin, G.J., 2004. The Theory and Practice of Revenue Management. Kluwer Academic Publishers, Boston.

Thompson, H.R., 1961. Statistical problems in airline reservation control. Operations Research Quarterly 12, 167–185.